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Equivalence of simplicial localizations of closed model categories

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Abstract

We determine a necessary and sufficient condition for a functor between closed model categories to induce an equivalence of Dwyer–Kan simplicial localizations. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

For a category $\mathscr{C}$ with a collection of weak equivalences $\mathscr{W}$, a basic object of study in homological algebra and in homotopy theory is the "localization" $\mathscr{C}[\mathscr{W}^{-1}]$. In [2], Dwyer and Kan show that $\mathscr{C}[\mathscr{W}^{-1}]$ is the "category of components" of a simplicial category $L\mathscr{C}$, the "simplicial localization". We think of the simplicial category $L\mathscr{C}$ as containing the "higher homotopy" information. If $\mathscr{C}$ is the category of spaces or simplicial sets and $\mathscr{W}$ is the collection of weak homotopy equivalences, then for nice X and Y, $L\mathscr{C}(X,Y)$ has the weak homotopy type of the mapping space Y^X. If $\mathscr{C}$ is the category of chain complexes of an abelian category with enough injectives or projectives and $\mathscr{W}$ is the collection of quasi-isomorphisms, then for nice X and Y, $\pi_*L\mathscr{C}(X,Y)$ is canonically isomorphic to the homology of the function complex $\mathrm{Hom}(X,Y)$ truncated at 0. More generally, if $\mathscr{C}$ is a "simplicial model category", then when X is cofibrant and Y is fibrant, $L\mathscr{C}(X,Y)$ has the weak homotopy type of the Hom simplicial set $\mathscr{C}_\bullet(X,Y)$; see [4] for details.

We study the question of when a functor induces an equivalence of this higher homotopy information. Precisely, for a functor $F:\mathscr{C}\to\mathscr{D}$ that takes all the weak equivalences of $\mathscr{C}$ to weak equivalences in $\mathscr{D}$, we consider the induced simplicial functor $LF:L\mathscr{C}\to L\mathscr{D}$ and ask when it is a "weak equivalence" [4, 2.4] of simplicial

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categories. For LF to be a weak equivalence in this case just means that the localization $\mathscr{C}[\mathscr{W}_{\mathscr{C}}^{-1}] \to \mathscr{D}[\mathscr{W}_{\mathscr{D}}^{-1}]$ is an equivalence of categories and that for all objects X and Y of $\mathscr{C}$, the map $L\mathscr{C}(X,Y) \to L\mathscr{D}(FX,FY)$ is a weak equivalence of simplicial sets.

For the statement of the main theorem, we shall need to restrict to the case when the categories in question are closed model categories. However, we do not assume that the functor considered preserves the model structure. Most often the functors of interest preserve only certain weak equivalences, for example weak equivalences between cofibrant objects or weak equivalences between fibrant objects. In our main result, we consider functors that preserve weak equivalences between objects that are both cofibrant and fibrant. In fact, we only need F to be defined on the full subcategory $\mathscr{M}_{cf}$ of cofibrant fibrant objects of a closed model category $\mathscr{M}$. It is shown in [3, 8.4] that the inclusion of $\mathscr{M}_{cf}$ in $\mathscr{M}$ induces a weak equivalence of simplicial categories $L\mathscr{M}_{cf} \to L\mathscr{M}$.

When $\mathscr{M}$ and $\mathscr{N}$ are closed model categories the localizations $\mathscr{M}[\mathscr{W}_{\mathscr{M}}^{-1}]$ and $\mathscr{N}[\mathscr{W}_{\mathscr{N}}^{-1}]$ are traditionally denoted as $\mathrm{Ho}\,\mathscr{M}$ and $\mathrm{Ho}\,\mathscr{N}$ and called homotopy categories [8]. A functor $F : \mathscr{M}_{cf} \to \mathscr{N}$ that preserves weak equivalences induces a functor $\mathrm{Ho}\,\mathscr{M} \to \mathrm{Ho}\,\mathscr{N}$, well-defined up to canonical natural isomorphism. If A is an object of $\mathscr{M}_{cf}$, then the category of objects under A, $\mathscr{M}\backslash A$, and the category of objects over A, $\mathscr{M}/A$, are closed model categories with the property that the cofibrant fibrant objects of $\mathscr{M}\backslash A$ and $\mathscr{M}\backslash A$ are also cofibrant fibrant objects of $\mathscr{M}$. Thus, F "restricts" to functors $(\mathscr{M}\backslash A)_{cf} \to \mathscr{N}\backslash FA$ and $(\mathscr{M}/A)_{cf} \to \mathscr{N}/FA$. Unless $\mathscr{N}$ is proper or FA happens to be cofibrant or fibrant, the category $\mathscr{N}\backslash FA$ or $\mathscr{N}/FA$ may not be the "right" category to consider. For a cofibrant approximation $A' \to FA$ and a fibrant approximation $FA \to A''$, the categories $\mathscr{N}\backslash A'$ and $\mathscr{N}/A''$ are better, and we can consider the restriction of F to functors $(\mathscr{M}\backslash A)_{cf} \to \mathscr{N}\backslash A'$ and $(\mathscr{M}/A)_{cf} \to \mathscr{N}/A''$. These functors preserve weak equivalences, and so we have induced functors $\mathrm{Ho}(\mathscr{M}\backslash A) \to \mathrm{Ho}(\mathscr{N}\backslash A')$ and $\mathrm{Ho}(\mathscr{M}/A) \to \mathrm{Ho}(\mathscr{N}/A'')$, well-defined up to canonical natural isomorphism. We can now state the main result.

Theorem 1.1. *Let $\mathscr{M}$ and $\mathscr{N}$ be closed model categories. Let $F : \mathscr{M}_{cf} \to \mathscr{N}$ be a functor that preserves weak equivalences and that induces an equivalence of homotopy categories $\mathrm{Ho}\,\mathscr{M} \to \mathrm{Ho}\,\mathscr{N}$. The following are equivalent.*

(1) *F induces a weak equivalence of simplicial categories $L\mathscr{M}_{cf} \to L\mathscr{N}$.*

(2) *For every object A of $\mathscr{M}_{cf}$, there is a cofibrant approximation $A' \to FA$ such that the induced functor $\mathrm{Ho}(\mathscr{M}\backslash A) \to \mathrm{Ho}(\mathscr{N}\backslash A')$ is fully faithful.*

(3) *For every object A of $\mathscr{M}_{cf}$ and every cofibrant approximation $A' \to FA$, the induced functor $\mathrm{Ho}(\mathscr{M}\backslash A) \to \mathrm{Ho}(\mathscr{N}\backslash A')$ is an equivalence.*

(4) *For every object B of $\mathscr{M}_{cf}$, there is a fibrant approximation $FB \to B'$ such that the induced functor $\mathrm{Ho}(\mathscr{M}/B) \to \mathrm{Ho}(\mathscr{N}/B')$ is fully faithful.*

(5) *For every object B of $\mathscr{M}_{cf}$ and every fibrant approximation $FB \to B'$, the induced functor $\mathrm{Ho}(\mathscr{M}/B) \to \mathrm{Ho}(\mathscr{N}/B')$ is an equivalence.*

Note that when F is the restriction of part of an adjoint pair satisfying Quillen's conditions [8, p. 4.6] (see also [5, 9.7.(i)–(ii)]), it is known that LF induces an equivalence of simplicial localizations [4, 1.1.(ii)]. Likewise, in this situation, it is easy to see that F satisfies (3) and (5) above. However, it has not been previously recognized that these properties of "Quillen equivalences" are closely related. Although Quillen equivalences comprise many examples that arise in practice, for functors that are not adjoints, the equivalence of (1) and (3) and the surprising equivalence of (3) and (5) were previously unknown.

As an application of the previous theorem, we offer a result on homotopy pushouts and homotopy pullbacks. In a closed model category, homotopy pushouts and homotopy pullbacks can be described in terms of coproducts in under-categories and pullbacks in over-categories. The following corollary is then an immediate consequence of the previous theorem.

Corollary 1.2. *Let $\mathscr{M}$ and $\mathscr{N}$ be closed model categories. If $F:\mathscr{M}_{cf}\to\mathscr{N}$ preserves weak equivalences and induces a weak equivalence $L\mathscr{M}_{cf}\to L\mathscr{N}$, then F preserves homotopy pushouts and homotopy pullbacks.*

Remark on set theoretic issues. The Dwyer–Kan simplicial localization $L\mathscr{C}$ of a category $\mathscr{C}$ has small Hom simplicial sets generally only when the category $\mathscr{C}$ is a small category. Since our arguments involve constructions employing only finite or countable limits or colimits, we have no set-theoretic problems in considering simplicial "sets" which are not small. When $\mathscr{C}$ is a closed model category, the simplicial Hom sets of $L\mathscr{C}$ are "homotopically small" [4, Section 2,4.1]. It can be checked without difficulty that our arguments never leave the context of homotopically small simplicial sets.

2. Two reductions for Theorem 1.1

In this section we give two reductions of the main theorem. The first reduction is to replace the "functor" L of [2] with the "functor" L^H of [3]. The second cuts our work in half by taking advantage of the dual nature of the statements (2), (3) and (4), (5) in Theorem 1.1

We have stated Theorem 1.1 in terms of the standard simplicial localization of [2] since this simplicial localization is philosophically the most basic: it is the cotriple derived "functor" of localization. On the other hand, the hammock localization L^H of [3] has nicer properties. Paramount among these is the relationship of $L\mathscr{C}$ and $L^H\mathscr{C}$ to the category $\mathscr{C}$. There are natural and canonical "inclusions" $\mathscr{C}\to L\mathscr{C}$ and $\mathscr{C}\to L^H\mathscr{C}$, but the "inclusion" of $\mathscr{C}$ into $L\mathscr{C}$ is a map of "graphs" and is not a functor. The inclusion of $\mathscr{C}$ in $L^H\mathscr{C}$ is a functor. The upshot of this for our work is that in using standard simplicial localization, the difference between composition in $\mathscr{C}$ and composition in $L\mathscr{C}$ introduces homotopies, but using the hammock localization eliminates these homotopies. In other words, for maps $f:X\to Y$ and $g:Y\to Z$ in a category $\mathscr{C}$, whereas the diagram on the left commutes only up to homotopy, the diagram on the right actually commutes.

$$\begin{tikzcd} L\mathscr{C}(W,X) \arrow[r, "f"] \arrow[dr, "g\circ f"'] & L\mathscr{C}(W,Y) \arrow[d, "g"] \\ & L\mathscr{C}(W,Z) \end{tikzcd} \qquad \begin{tikzcd} L^H\mathscr{C}(W,X) \arrow[r, "f"] \arrow[dr, "g\circ f"'] & L^H\mathscr{C}(W,Y) \arrow[d, "g"] \\ & L^H\mathscr{C}(W,Z) \end{tikzcd}$$

On the other hand, by [3, 2.2], the hammock localization and the standard simplicial localizations are related by a functorial chain of weak equivalences. It follows that a functor induces a weak equivalence on the standard simplicial localization if and only if it induces an equivalence on the hammock localization. Thus, we have the following reduction for Theorem 1.1

Reduction 1. *It suffices to prove the analogue of Theorem* 1.1 *for the hammock localization* L^H.

Because of the better naturality properties of the hammock localization, we shall use it exclusively in the remainder of the paper, denoting it by L instead of L^H. *Henceforth the notation* $L\mathscr{C}$ *will denote the hammock localization of the category* $\mathscr{C}$ *and not the standard simplicial localization.*

For the second reduction, we note that condition (4) is equivalent to condition (2) applied to the opposite functor $F^{\mathrm{op}}: \mathscr{M}_{cf}^{\mathrm{op}} \to \mathscr{N}^{\mathrm{op}}$. Similarly, condition (5) is equivalent to condition (3) for F^{op}. In addition, since we have canonical isomorphisms of simplicial categories $L(\mathscr{M}_{cf}^{\mathrm{op}}) \cong (L\mathscr{M}_{cf})^{\mathrm{op}}$ and $L(\mathscr{N}^{\mathrm{op}}) \cong (L\mathscr{N})^{\mathrm{op}}$, the functor LF is a weak equivalence of simplicial categories if and only if the functor LF^{op} is. Thus, we have the following reduction for Theorem 1.1.

Reduction 2. *It suffices to show the equivalence of* (1), (2), *and* (3).

3. The equivalence of (2) and (3)

The equivalence of (2) and (3) in the main theorem is a straightforward closed model category theory argument. The proof breaks down into two steps. First we need to change the “for some” into a “for every”. This is accomplished by the following proposition.

Proposition 3.1. *Let* $A_1' \to FA$ *and* $A_2' \to FA$ *be cofibrant approximations. If the induced functor* $\mathrm{Ho}(\mathscr{M}\backslash A) \to \mathrm{Ho}(\mathscr{N}\backslash A_1')$ *is fully faithful then so is the induced functor* $\mathrm{Ho}(\mathscr{M}\backslash A) \to \mathrm{Ho}(\mathscr{N}\backslash A_2')$.

Proof. Let A' be a cofibrant approximation of the pullback $A_1' \times_{FA} A_2'$. The maps $A' \to A_1'$ and $A' \to A_2'$ are acyclic fibrations and in particular weak equivalences. We obtain "forgetful" functors $\mathscr{N}\backslash A'_1 \to \mathscr{N}\backslash A'$ and $\mathscr{N}\backslash A'_2 \to \mathscr{N}\backslash A'$ that preserve weak equivalences and therefore induce functors on the homotopy categories. The composite functors $\mathrm{Ho}(\mathscr{M}\backslash A) \to \mathrm{Ho}(\mathscr{N}\backslash A')$ coincide. The proof is then completed by the following proposition. □

Proposition 3.2. *Let $X \to Y$ be a weak equivalence between cofibrant objects. The functor $\mathscr{N}\backslash Y \to \mathscr{N}\backslash X$ is a Quillen equivalence and therefore induces an equivalence of homotopy categories.*

Proof. The functor $R : \mathscr{N}\backslash Y \to \mathscr{N}\backslash X$ has as left adjoint the functor L that takes an object $X \to Z$ of $\mathscr{N}\backslash X$ to the object $Y \to Z \amalg_X Y$ of $\mathscr{N}\backslash Y$. This adjunction satisfies the condition [5, 9.7.(i)] that ensures that the derived functors exist and give an adjunction of homotopy categories. To check condition [5, 9.7.(ii)], it suffices to see that for a cofibration $X \to Z$, the induced map $Z \to Z \amalg_X Y$ is a weak equivalence in $\mathscr{N}$. When the map $X \to Y$ is an acyclic cofibration, this is clear from the model category axioms. When the map $X \to Y$ is an arbitrary weak equivalence, this is proved using the argument [5, 9.9] of K. Brown as follows.

Consider the map $X \amalg Y \to Y$ induced by the given map $X \to Y$ and the identity map of Y. Factor this as a cofibration and an acyclic fibration $X \amalg Y \to W \to Y$. Then the maps $X \to W$ and $Y \to W$ are acyclic cofibrations and therefore induce weak equivalences

$$Z \to Z \amalg_X W \quad \text{and} \quad Z \amalg_X Y \to (Z \amalg_X Y) \amalg_Y W = Z \amalg_X W.$$

Since the composite $Y \to W \to Y$ is the identity on Y, the map

$$Z \amalg_X W \to (Z \amalg_X W) \amalg_W Y = Z \amalg_X Y$$

is a weak equivalence. The map we are interested in coincides with the composite of weak equivalences

$$Z \to Z \amalg_X W \to (Z \amalg_X W) \amalg_W Y = Z \amalg_X Y$$

and is therefore a weak equivalence. □

Finally we need to show that if F induces an equivalence $\mathrm{Ho}\,\mathscr{M} \to \mathrm{Ho}\,\mathscr{N}$, the induced functor $\mathrm{Ho}(\mathscr{M}\backslash A) \to \mathrm{Ho}(\mathscr{N}\backslash A')$ is an equivalence whenever it is full and faithful. This is a consequence of the following proposition.

Proposition 3.3. *If F induces an equivalence* $\mathrm{Ho}\,\mathscr{M} \to \mathrm{Ho}\,\mathscr{N}$ *then every object of* $\mathrm{Ho}(\mathscr{N}\backslash A')$ *is isomorphic to one in the image of* $\mathrm{Ho}(\mathscr{M}\backslash A)$.

Proof. The proof is the usual "mapping cylinder" argument. Let $\mathbf{X}$ be an object in $\mathscr{N}\backslash A'$ with underlying object X in $\mathscr{N}$. It suffices to consider the case when $\mathbf{X}$ is cofibrant. Since $\mathrm{Ho}\,\mathscr{M} \to \mathrm{Ho}\,\mathscr{N}$ is an equivalence, we can find an object $\mathbf{Y}$ in $\mathscr{M}\backslash A$ such that $F\mathbf{Y}$ is isomorphic to $\mathbf{X}$ in $(\mathrm{Ho}\,\mathscr{N})\backslash A'$. Write Y for the underlying object in $\mathscr{M}$. Choose a fibrant approximation $\varepsilon : FY \to Y'$. Then we can find a weak equivalence $\phi : X \to Y'$ such that the composite maps $f : A' \to X \to Y'$ and $g : A' \to FY \to Y'$ represent the same map in $\mathrm{Ho}\,\mathscr{N}$. It follows that there exists a left cylinder object IA' of A' and a homotopy $h : IA' \to Y'$ such that $h \circ i_0 = f$ and $h \circ i_1 = g$, where i_0 and i_1 are the face maps. Let Z be $IA' \amalg_{A'} X$, the pushout of IA' and X under i_0 and regard Z as an object in $\mathscr{N}\backslash A'$ via the map i_1.

The map h together with ϕ induces a map under A' from Z to Y' and the collapse map on IA' together with id_X induces a map under A' from Z to X. Since i_0 is an acyclic cofibration, the inclusion of X in the pushout Z is an acyclic cofibration. The composite of this map with the maps $Z \to Y'$ and $Z \to X$ are ϕ and id_X respectively, and it follows that the maps $Z \to Y'$ and $Z \to X$ are weak equivalences. These maps together with ε give an isomorphism in $\mathrm{Ho}(\mathscr{N}\backslash A')$ between $\mathbf{X}$ and $F\mathbf{Y}$. $\square$

4. The implication (1) ⇒ (2)

In this section, we prove that (1) implies (2) in Theorem 1.1. We use a lemma that explains the relationship between the simplicial localization of a closed model category and the simplicial localization of its under categories. The lemma itself is proved in Section 6.

In the following lemma, $\mathscr{M}$ denotes a closed model category, A an object in $\mathscr{M}$, and $\mathbf{X}$ and $\mathbf{Y}$ objects in $\mathscr{M}\backslash A$. We denote by X and Y the underlying objects in $\mathscr{M}$ of $\mathbf{X}$ and $\mathbf{Y}$. We denote the initial object in $\mathscr{M}\backslash A$ (the identity map of A) as $\mathbf{A}$.

Lemma 4.1. *When A is cofibrant, the square*

$$\begin{array}{ccc} L(\mathscr{M}\backslash A)(\mathbf{X},\mathbf{Y}) & \longrightarrow & L(\mathscr{M}\backslash A)(\mathbf{A},\mathbf{Y}) \\ \downarrow & & \downarrow \\ L\mathscr{M}(X,Y) & \longrightarrow & L\mathscr{M}(A,Y) \end{array}$$

is a homotopy pullback square of simplicial sets.

Since the inclusion of $\mathscr{M}_{cf}$ into $\mathscr{M}$ induces a weak equivalence on simplicial localizations, it follows that when $\mathbf{X}$ and $\mathbf{Y}$ are cofibrant fibrant in $\mathscr{M}\backslash A$, the square

$$\begin{array}{ccc} L(\mathscr{M}\backslash A)_{cf}(\mathbf{X},\mathbf{Y}) & \longrightarrow & L(\mathscr{M}\backslash A)_{cf}(\mathbf{A},\mathbf{Y}) \\ \downarrow & & \downarrow \\ L\mathscr{M}_{cf}(X,Y) & \longrightarrow & L\mathscr{M}_{cf}(A,Y) \end{array}$$

is also a homotopy pullback square of simplicial sets.

Now let $\mathscr{N}$ be a closed model category, and let $F:\mathscr{M}_{cf}\to\mathscr{N}$ be a functor that preserves weak equivalences. Let A be a cofibrant fibrant object of $\mathscr{M}$, and let $\eta:A'\to FA$ be a cofibrant approximation. Write $\mathbf{A}'$ for the initial object of $\mathscr{N}\backslash A'$. We obtain a commutative diagram

$$\begin{tikzcd}
L(\mathscr{M}\backslash A)_{cf}(\mathbf{X},\mathbf{Y}) \arrow[rr] \arrow[dr] \arrow[dd] & & L(\mathscr{M}\backslash A)_{cf}(\mathbf{A},\mathbf{Y}) \arrow[dr] \arrow[dd] & \\
& L(\mathscr{N}\backslash A')(F\mathbf{X},F\mathbf{Y}) \arrow[rr] \arrow[dd] & & L(\mathscr{N}\backslash A')(\mathbf{A}',F\mathbf{Y}) \arrow[dd] \\
L\mathscr{M}_{cf}(X,Y) \arrow[rr] \arrow[dr] & & L\mathscr{M}_{cf}(A,Y) \arrow[dr] & \\
& L\mathscr{N}(FX,FY) \arrow[rr] & & L\mathscr{N}(A',FY).
\end{tikzcd}$$

Since the top and bottom squares are both homotopy pullbacks by the lemma, the implication $(1)\Rightarrow(2)$ is an immediate consequence of the following proposition.

Proposition 4.2. *The simplicial sets* $L(\mathscr{M}\backslash A)_{cf}(\mathbf{A},\mathbf{Y})$ *and* $L(\mathscr{N}\backslash A')(\mathbf{A}',F\mathbf{Y})$ *are weakly contractible.*

Proof. In [3, 4], it was shown that for any closed model category $\mathscr{C}$, any cofibrant object U of $\mathscr{C}$, and any object V of $\mathscr{C}$, there is a simplicial object $V_\bullet$ of $\mathscr{C}$ such that the simplicial set of maps $\mathscr{C}(U,V_\bullet)$ is weakly equivalent to $L\mathscr{C}(U,V)$. We plug in $\mathscr{M}\backslash A$ or $\mathscr{N}\backslash A'$ for $\mathscr{C}$, $\mathbf{A}$ or $\mathbf{A}'$ for U, and $\mathbf{Y}$ or $F\mathbf{Y}$ for V. Since U is an initial object, $\mathscr{C}(U,V_\bullet)$ is the one point simplicial set. We conclude that $L(\mathscr{M}\backslash A)(\mathbf{A},\mathbf{Y})$ and $L(\mathscr{N}\backslash A')(\mathbf{A}',F\mathbf{Y})$ are weakly contractible. Finally $L(\mathscr{M}\backslash A)_{cf}(\mathbf{A},\mathbf{Y})$ is weakly contractible since it is weakly equivalent to $L(\mathscr{M}\backslash A)(\mathbf{A},\mathbf{Y})$. □

5. The implication (3) ⇒ (1)

In this section, we use Lemma 4.1 to identify the iterated loop spaces of the mapping complex $L\mathscr{M}(X,Y)$ as the mapping complexes in certain undercategories. Since $\pi_0 L(\mathscr{M}\backslash A)(X,Y)$ is the set of maps from X to Y in the homotopy category $\mathrm{Ho}(\mathscr{M}\backslash A)$, this gives an interpretation of the higher homotopy groups of $L\mathscr{M}(X,Y)$ in terms of the homotopy categories of under-categories. We use this to prove the implication $(3)\Rightarrow(1)$ at the end of this section.

To construct the loop space at a given base point, the under-category we need to consider is $\mathscr{M}\backslash X\amalg X$. The following lemma is proved in the next section.

Lemma 5.1. *If X is a cofibrant object of $\mathscr{M}$, then the map $L\mathscr{M}(X \amalg X, Y) \to L\mathscr{M}(X, Y) \times L\mathscr{M}(X, Y)$ induced by the two inclusions of X in $X \amalg X$ is a weak equivalence.*

Now let X be cofibrant and let Y be fibrant; then for each component of $L\mathscr{M}(X, Y)$, we can choose for the base point a zero simplex determined by a map $f : X \to Y$. Write $\mathbf{X}$ for the object $X \amalg X \to X$ of $\mathscr{M} \backslash X \amalg X$ corresponding to the codiagonal map, and write $\mathbf{Y}$ for the object $X \amalg X \to Y$ corresponding to the composite of the codiagonal with the map f. Applying Lemma 4.1, Lemma 5.1, and Proposition 4.2, we obtain the following result.

Proposition 5.2. *In the notation and with the assumptions above, the based simplicial set $L(\mathscr{M} \backslash X \amalg X)(\mathbf{X}, \mathbf{Y})$ is weakly equivalent to $\Omega_f L\mathscr{M}(X, Y)$.*

Here we understand $L(\mathscr{M} \backslash X \amalg X)(\mathbf{X}, \mathbf{Y})$ to be based at the map $f : \mathbf{X} \to \mathbf{Y}$. Taking a cofibrant approximation of $\mathbf{X}$ in $\mathscr{M} \backslash X \amalg X$, we can apply Proposition 5.2 with the closed model category $\mathscr{M} \backslash X \amalg X$ in the place of $\mathscr{M}$. To iterate Proposition 5.2 further, we introduce the following notation.

Definition 5.3. Let $X \times S^0$ be $X \amalg X$. Inductively, let $X \times B^n$ be an object with a weak equivalence $X \times B^n \to X$ and a cofibration $X \times S^{n-1} \to X \times B^n$ over X. Let $X \times S^n$ be the coproduct of two copies of $X \times B^n$ in the category $\mathscr{M} \backslash (X \times S^{n-1}), (X \times B^n) \amalg_{(X \times S^{n-1})} (X \times B^n)$.

Iterating Proposition 5.2 we obtain the following proposition.

Proposition 5.4. *The based simplicial set $L(\mathscr{M} \backslash (X \times S^{n-1}))(X \times B^n, Y)$ is weakly equivalent to $\Omega_f^n L\mathscr{M}(X, Y)$.*

Now assume that X and Y are cofibrant fibrant. In order to prove the implication (3) ⇒ (1), we need to understand the naturality of the previous proposition in functors $F : \mathscr{M}_{cf} \to \mathscr{N}$. This is complicated by the fact that the objects $X \times S^n$ are cofibrant but not necessarily fibrant. To fix this, factor the map $X \times S^0 \to X$ through an acyclic cofibration $X \times S^0 \to Q^0$ and a fibration $Q^0 \to X$. Inductively, let

$$C^n = Q^{n-1} \amalg_{(X \times S^{n-1})} (X \times B^n),$$

and factor the induced map $C^n \to X$ as an acyclic cofibration $C^n \to D^n$ and a fibration $D^n \to X$. Then the map $Q^{n-1} \to D^n$ is a cofibration, and the map $X \times B^n \to D^n$ is an acyclic cofibration that factors the map $X \times B^n \to X$. Let $R^n = D^n \amalg_{Q^{n-1}} D^n$. The induced map $X \times S^n \to R^n$ is an acyclic cofibration. Factor the map $R^n \to X$ as an acyclic cofibration $R^n \to Q^n$ followed by a fibration $Q^n \to X$.

In other words, we have constructed cofibrant fibrant objects Q^{n-1}, D^n over X, acyclic cofibrations $X \times S^{n-1} \to Q^{n-1}$, $X \times B^n \to D^n$ over X, and a cofibration

$Q^{n-1} \to D^n$ over X that makes the following diagram commute.

$$\begin{tikzcd} X \times S^{n-1} \arrow[r] \arrow[d] & Q^{n-1} \arrow[d] \\ X \times B^n \arrow[r] & D^n \end{tikzcd}$$

Proposition 3.2 and [4, 1.1.(ii)] imply the following result.

Proposition 5.5. *The maps $X \times S^{n-1} \to Q^{n-1}$ and $X \times B^n \to D^n$ induce a weak equivalence $L(\mathscr{M}\backslash Q^{n-1})(D^n, Y) \to L(\mathscr{M}\backslash(X \times S^{n-1}))(X \times B^n, Y)$.*

Let $F : \mathscr{M}_{cf} \to \mathscr{N}$ be a functor that preserves weak equivalences, and let $X' \to FX$ be a cofibrant approximation. The map $X \amalg X \to Q^0$ induces a map $FX \amalg FX \to FQ^0$ and therefore a map $X' \amalg X' \to FQ^0$ that makes the following diagram commute.

$$\begin{tikzcd} X' \amalg X' \arrow[r] \arrow[d] & FQ^0 \arrow[d] \\ X' \arrow[r] & FX \end{tikzcd}$$

Let f' be the composite $X' \to FX \to FY$.

Let $X' \times S^0 = X' \amalg X'$. Inductively, factor the composite map $X' \times S^{n-1} \to FD^n$ through a cofibration $X' \times S^{n-1} \to X' \times B^n$ followed by an acyclic fibration $X' \times B^n \to FD^n$. The lifting property for acyclic fibrations allows us to choose a lift in the following diagram

$$\begin{tikzcd} X' \times S^{n-1} \arrow[r] \arrow[d] & X' \arrow[d] \\ X' \times B^n \arrow[r] \arrow[ur, dotted] & FX \end{tikzcd}$$

Let $X' \times S^n$ be the coproduct of two copies of $X' \times B^n$ in $\mathscr{N}\backslash(X' \times S^{n-1})$, $(X' \times B^n) \amalg_{(X' \times S^{n-1})} (X' \times B^n)$. The map $D^n \amalg_{Q^{n-1}} D^n \to Q^n$ induces a map $FD^n \amalg_{FQ^{n-1}} FD^n \to FQ^n$ and therefore a map $X' \times S^n \to FQ^n$.

The objects $X' \times S^{n-1}$ and $X' \times B^n$ satisfy the analogue of Definition 5.3, and so the analogue of Proposition 5.4 holds in $\mathscr{N}$ for these objects. It now becomes a simple matter of keeping track of the chain of maps inducing the weak equivalence of Proposition 5.2 to prove the following proposition.

Proposition 5.6. *The following diagram in the homotopy category of simplicial sets commutes.*

$$\begin{tikzcd} L(\mathscr{M}\backslash Q^{n-1})_{cf}(D^n, Y) \arrow[r, "\sim"] \arrow[d, "LF"'] & \Omega^n_f L\mathscr{M}_{cf}(X,Y) \arrow[d, "\Omega LF"] \\ L(\mathscr{N}\backslash X' \times S^{n-1})(X' \times B^n, FY) \arrow[r, "\sim"'] & \Omega^n_{f'} L\mathscr{N}(X', FY) \end{tikzcd}$$

Now assume that F satisfies (3) in Theorem 1.1. Then in particular F preserves coproducts in the homotopy categories of the under-categories. Since $X \times S^n$ and $X' \times S^n$ represent the coproducts of two copies of X and X' in the homotopy categories $\mathrm{Ho}(\mathscr{M}\backslash(X \times S^{n-1}))$ and $\mathrm{Ho}(\mathscr{N}\backslash(X' \times S^{n-1}))$ respectively, we inductively conclude that the map $X' \times S^n \to FQ^n$ is a weak equivalence for all $n \geq 0$. Then F induces an equivalence between $\mathrm{Ho}(\mathscr{M}\backslash Q^n)$ and $\mathrm{Ho}(\mathscr{N}\backslash(X' \times S^n))$ and hence the map

$$\pi_0 L(\mathscr{M}\backslash Q^n)_{cf}(D^{n+1}, Y) \to \pi_0 L(\mathscr{N}\backslash X' \times S^n)(X' \times B^{n+1}, FY)$$

is a bijection. We conclude from Proposition 5.6 that LF induces a bijection

$$\pi_{n+1} L\mathscr{M}_{cf}(X,Y)_f \to \pi_{n+1} L\mathscr{N}(X', FY)_{f'}$$

for all $n \geq 0$. Since f was arbitrary and LF induces a bijection on path components, the map $LF : L\mathscr{M}_{cf}(X,Y) \to L\mathscr{N}(FX, FY)$ is a weak equivalence.

Remark 5.7. The objects $X \times S^{n-1}$ and $X \times B^n$ of Definition 5.3 can be thought of as higher cylinder objects and their boundaries, and can be used (together with a diagramatic generalization) to define higher left homotopies in a model category. Proposition 5.4 then gives a bijection between the homotopy groups of the simplicial localization and the set of higher homotopies of a map.

6. The proofs of Lemmas 4.1 and 5.1

In this section we prove Lemmas 4.1 and 5.1. The proofs rely on the equivalence of the simplicial function complex $L\mathscr{M}(X,Y)$ with the simplicial set of maps $\mathscr{M}(X, Y_\bullet)$ for a simplicial resolution $Y \to Y_\bullet$ [4, 4.3]. In order to make effective use of this, we need to understand the precise relationship between these two simplicial sets. We do this with the following lemma, proved in Section 8.

Lemma 6.1. *Let X be cofibrant and let $Y \to Y_\bullet$ be a simplicial resolution. Then the inclusions*

$$\mathscr{M}(X, Y_\bullet) \to \operatorname{diag} L\mathscr{M}(X, Y_\bullet) \leftarrow L\mathscr{M}(X,Y)$$

are weak equivalences. The composite isomorphism in the homotopy category $\mathscr{M}(X, Y_\bullet) \sim L\mathscr{M}(X, Y)$ *is the one constructed in* [4].

Lemma 5.1 is an easy consequence.

Proof of Lemma 5.1. The weak equivalences in Lemma 6.1 reduce this to showing that the map $\mathscr{M}(X \amalg X, Y_\bullet) \to \mathscr{M}(X, Y_\bullet) \times \mathscr{M}(X, Y_\bullet)$ is a weak equivalence. In fact, this map is an isomorphism. $\square$

Lemma 4.1 is proved as follows. By factoring the map $A \to X$, we can assume without loss of generality that X is cofibrant and the map $A \to X$ is a cofibration. By inspection, the square

$$\begin{array}{ccc} \mathscr{M}\backslash A(\mathbf{X}, \mathbf{Y}_\bullet) & \longrightarrow & \mathscr{M}\backslash A(\mathbf{A}, \mathbf{Y}_\bullet) \\ \downarrow & & \downarrow \\ \mathscr{M}(X, Y_\bullet) & \longrightarrow & \mathscr{M}(A, Y_\bullet) \end{array} \tag{6.2}$$

is a pullback square. If we show that the map $\mathscr{M}(X, Y_\bullet) \to \mathscr{M}(A, Y_\bullet)$ is a Kan fibration, then we can conclude that the square (6.2) is a homotopy pullback square, and Lemma 4.1 will follow from Lemma 6.1. Thus, the proof of Lemma 4.1 is completed by the following proposition.

Proposition 6.3. *The map* $\mathscr{M}(X, Y_\bullet) \to \mathscr{M}(A, Y_\bullet)$ *is a Kan fibration.*

The Kan condition translates into a lifting question on maps: we need to see that given a map $f: A \to Y_{n+1}$ and maps $g_i: X \to Y_n$ for $0 \le i < n$ compatible on the faces $d_j: Y_n \to Y_{n-1}$ and such that the restriction to A of g_i is $d_i \circ f$, we can find a map $h: X \to Y_{n+1}$ that restricts to f on A. Clearly, by taking a certain limit, we can rephrase this as a lifting problem of the usual form [5, 3.2]; this is how we proceed.

For $n \ge 0$ and $0 \le k \le n+1$, let $\mathbf{L}_n^k$ be the diagram in $\mathscr{M}$ consisting of objects:

- for each i, $0 \le i \le k$, a copy of Y_n labeled (d_i, Y_n);
- for each (i,j), $0 \le i < j \le k$, a copy of Y_{n-1} labeled $(d_i d_j, Y_{n-1})$; (We understand Y_{-1} as the final object.)

and maps

- for each (i,j), $0 \le i < j \le k$, a map $(d_j, Y_n) \to (d_i d_j, Y_{n-1})$ given by the map $d_i: Y_n \to Y_{n-1}$;
- for each (i,j), $0 \le i < j \le k$, a map $(d_j, Y_n) \to (d_i d_j, Y_{n-1})$ given by the map $d_{j-1}: Y_n \to Y_{n-1}$.

Let $L_n^k = \operatorname{Lim} \mathbf{L}_n^k$.

The face maps $d_0, \ldots, d_k$ on Y_{n+1} induce a map $Y_{n+1} \to L_n^k$. In the case $k = n+1$, L_n^{n+1} is the object called (d_*, Y_n) in [4, 4.3], and one of the conditions for $Y_\bullet$ to be a simplicial resolution is for the map $Y_{n+1} \to L_n^{n+1}$ to be a fibration. For Proposition 6.3, we are interested in the case $k=n$. The compatible maps $g_i : X \to Y_n$ induce a map $g : X \to L_n^n$, and the lifting problem described above then translates to finding a lift for the following square.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & Y_{n+1} \arrow[d, "d_0 \times \cdots \times d_n"] \\ X \arrow[r, "g"'] & L_n^n \end{tikzcd}$$

Since $A \to X$ is a cofibration, Proposition 6.3 is an immediate consequence of the following proposition in the case $k=n$.

Proposition 6.4. *The map $Y_{n+1} \to L_n^k$ is an acyclic fibration for $0 \le k \le n$.*

Proof. We prove this by induction on n. In the base case $n=0$, L_0^0 is Y_0 and the map is the zeroth face map $d_0 : Y_1 \to Y_0$, which is an acyclic fibration by assumption [4, 4.3.(ii)].

For $0 \le k \le n$, consider the following two subdiagrams $\mathbf{D}$ and $\mathbf{E}$ of $\mathbf{L}_n^{k+1}$. Let $\mathbf{D}$ be the full subdiagram consisting of the (d_{k+1}, Y_n)'s and the $(d_i d_{k+1}, Y_n)$'s for all i. Let $\mathbf{E}$ be the full subdiagram consisting of the (d_i, Y_n)'s for $i < k+1$ and all $(d_i d_j, Y_n)$'s. Then the diagram $\mathbf{L}_n^{k+1}$ is the "union" of $\mathbf{D}$ and $\mathbf{E}$ in the sense that every map in $\mathbf{L}_n^{k+1}$ is either a map in $\mathbf{D}$ or a map in $\mathbf{E}$. This allows us to write the limit L_n^{k+1} as the pullback of the limits Lim $\mathbf{D}$ and Lim $\mathbf{E}$ over the limit of $\mathbf{D} \cap \mathbf{E}$. The diagram consisting of the object (d_{k+1}, Y_n) is an initial subdiagram of $\mathbf{D}$, and the diagram $\mathbf{L}_n^k$ is an initial subdiagram of $\mathbf{E}$. From this we see that the diagram on the left below is a pullback.

$$\begin{tikzcd} L_n^{k+1} \arrow[r] \arrow[d] & (d_{k+1}, Y_n) \arrow[d, "\times_i d_i"] \\ L_n^k \arrow[r, "d_k"'] & \times_i (d_i d_{k+1}, Y_{n-1}) \end{tikzcd} \qquad \begin{tikzcd} L_n^{k+1} \arrow[r] \arrow[d] & Y_n \arrow[d, "\times_i d_i"] \\ L_n^k \arrow[r, "d_k"'] & L_{n-1}^k \end{tikzcd}$$

Since as objects of $\mathscr{M}$, $(d_i d_{k+1}, Y_{n-1}) = (d_i, Y_{n-1})$, we can identify the product $\times_i (d_i d_{k+1}, Y_{n-1})$ with $\times (d_i, Y_{n-1})$. The simplicial relations among the face maps imply that both maps, $Y_n \to \times_i (d_i, Y_{n-1})$ and $L_n^k \to \times_i (d_i, Y_{n-1})$, factor through the subobject L_{n-1}^k. Thus, we see that the diagram to the right above is a pullback. The map on the right in this diagram is easily seen to be the face map $Y_n \to L^k_{n-1}$ and is therefore an acyclic fibration by induction when $k<n$. When $k=n$, this map is a fibration by the resolution assumption [4, 4.3.(iii)]. It follows that the map $L^{k+1}_n \to L^k_n$ is an acyclic fibration for $k<n$ and a fibration for $k=n$.

By induction on k, the map $Y_{n+1} \to L^k_n$ is a weak equivalence for $0 \le k \le n$. The base case $k=0$ follows from the resolution assumption [4, 4.3.(ii)]. The inductive step follows from the fact that the map $Y_{n+1} \to L^k_n$ factors through the map $L^{k+1}_n \to L^k_n$. By downward induction on k, we see that the map $Y_{n+1} \to L^k_n$ is a fibration for all k. The base case $Y_{n+1} \to L^{n+1}_n$ is the resolution assumption [4, 4.3.(iii)]. □

7. Review of the equivalence of diag $\mathscr{M}(X^\bullet, Z_\bullet)$ with $L\mathscr{M}(X,Z)$

The argument for Lemma 6.1 in the next section proceeds by comparison maps based on the chain of maps given in [4, 7.2]. These arguments require the use of fine details of these maps, and because these are complicated, we review them in this section.

Let X be cofibrant and Z be fibrant, and choose a special cosimplicial resolution $X^\bullet \to X$ and a special simplicial resolution $Z \to Z_\bullet$. We can choose $X^\bullet$ with $X^0 = X$ and the map $X^0 \to X$ the identity, and similarly, we can choose the map $Z \to Z_0$ to be the identity. Consider the bisimplicial set $\mathscr{M}(Z^\bullet, Z_\bullet)$. The basic statement of [4, 7.2] is that there exists a simplicial set $M_\bullet$ and weak equivalences

$$\operatorname{diag} \mathscr{M}(X_\bullet, Z_\bullet) \xleftarrow{\sim} M_\bullet \xrightarrow{\sim} L\mathscr{M}(X,Z).$$

Let $M_\bullet$ be the simplicial set whose m-simplices are the diagrams

$$\begin{tikzcd} X^{p_o} \arrow[r, "X(f_1)"] & \cdots \arrow[r, "X(f_m)"] & X^{p_m} \arrow[dll, "h"'] \\ Z_{q_o} \arrow[r, "Z(g_1)"'] & \cdots \arrow[r, "Z(g_m)"'] & Z_{q_m} \end{tikzcd} \tag{7.1}$$

where the horizontal maps are induced by the cosimplicial structure maps of $X^\bullet$ and the simplicial structure maps of $Z_\bullet$ by the maps $f_1, \dots, f_m$ in Δ and $g_1, \dots, g_m$ in Δ^{op}. The diagonal map h is any map in $\mathscr{M}(X^{p_m}, Z_{q_0})$. We have a simplicial map $M_\bullet \to \operatorname{diag} \mathscr{M}(X^\bullet, Z_\bullet)$ that takes the m-simplex (7.1) to the element $Z(g) \circ h \circ X(f)$ in $\mathscr{M}(X^m, Z_m)$ where f is the map in Δ that sends $(0, \dots, m)$ to $(f_1(p_0), \dots, f_m(p_m), p_m)$, and g is the map in Δ opposite to the map that takes $(0, \dots, m)$ to $(0, g_1^{\mathrm{op}}(0), \dots, g_m^{\mathrm{op}}(m))$.

Proposition 7.2. *The map $M_\bullet \to \operatorname{diag} \mathscr{M}(X^\bullet, Z_\bullet)$ is a weak equivalence.*

Proof. It suffices to see that the induced map on geometric realizations is a weak equivalence. The geometric realization of $M_\bullet$ is isomorphic to the geometric realization of the simplicial set $M'_\bullet$ whose m-simplices are the diagrams of the form

$$\begin{tikzcd} X^{p_0} \arrow[d, "h"'] & \cdots \arrow[l, "X(f_1)"'] & X^{p_m} \arrow[l, "X(f_m)"'] \\ Z_{q_0} \arrow[r, "Z(g_1)"'] & \cdots \arrow[r, "Z(g_m)"'] & Z_{q_m}. \end{tikzcd}$$

The simplicial set $M'_\bullet$ is the homotopy colimit of $\mathscr{M}(X^\bullet, Z_\bullet)$ viewed as a functor from the category $\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}$ to the category of simplicial sets. The composite map $|M'_\bullet| \to |\mathrm{diag}\, \mathscr{M}(X^\bullet, Z_\bullet)|$ is the geometric realization of the usual map from the homotopy colimit to the diagonal [1, XII.3.4], which is a weak equivalence [1, XII.4.3]. $\square$

Consider the category $\mathscr{C} = (\mathscr{W} \cap \mathscr{C}of)^{-1} \mathscr{M} (\mathscr{W} \cap \mathscr{F}ib)^{-1}(X, Z)$ of [4, 7.2]. This category has as objects the diagrams

$$X \leftarrow A \rightarrow B \leftarrow Z$$

where the map $A \to X$ is an acyclic fibration and the map $Z \to B$ is an acyclic cofibration. There is a canonical inclusion $N_\bullet \mathscr{C} \to L\mathscr{M}(X, Z)$ that according to [4, 7.2.(ii)] is a weak equivalence. We have a simplicial map $M_\bullet \to N_\bullet \mathscr{C}$ that takes the m-simplex pictured in (7.1) to the m-simplex of $N_\bullet \mathscr{C}$ specified by the diagram

$$\begin{tikzcd} & & X & & \\ X^{p_0} \arrow[urr] \arrow[r, "X(f_1)"] \arrow[d, "h \circ X(f_m \circ \cdots \circ f_1)"'] & \cdots \arrow[rr, "X(f_m)"] & & X^{p_m} \arrow[ul] \arrow[d, "Z(g_m \circ \cdots \circ g_1) \circ h"] & \\ Z_{q_0} \arrow[r, "Z(g_1)"] & \cdots \arrow[rr, "Z(g_m)"] & & Z_{q_m} & \\ & & Z. \arrow[ull] \arrow[ur] & & \end{tikzcd}$$

The assumption that $X^\bullet \to X$ and $Z \to Z_\bullet$ are special resolutions implies that the maps $X^{p_i} \to X$ are acyclic fibrations and the maps $Z \to Z_{q_i}$ are acyclic cofibrations.

Proposition 7.3. *The map $M_\bullet \to N_\bullet \mathscr{C}$ is a weak equivalence.*

Proof. (Cf. [4, 7.2.(iii)].) The map $M_\bullet \to N_\bullet \mathscr{C}$ factors through the simplicial set $N_\bullet$ whose m-simplices are the diagrams

$$\begin{tikzcd}
 & X & \\
A_0 \arrow[ur] \arrow[r] & \cdots \arrow[r] & A_m \arrow[ul] \arrow[dll] \\
B_0 \arrow[r] & \cdots \arrow[r] & B_m \\
 & Z \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

where the maps $A_i \to X$ are acyclic fibrations and the maps $Z \to B_i$ are acyclic cofibrations. Playing the same trick as in the proof of Proposition 7.2, we can identify $|N_\bullet|$ as the geometric realization of a homotopy colimit and use [4, 6.11–6.12] to conclude that the map $M_\bullet \to N_\bullet$ is a weak equivalence.

To see that the map $N_\bullet \to N_\bullet\mathscr{C}$ is a weak equivalence, consider the bisimplicial set $P_{\bullet\bullet}$ whose set of km-simplices consists of the diagrams

$$\begin{tikzcd}
 & X & & & & \\
A_0 \arrow[ur] \arrow[r] \arrow[d] & \cdots \arrow[r] & A_m \arrow[ul] \arrow[d] & & & \\
B_0 \arrow[r] & \cdots \arrow[r] & B_m \arrow[r] & C_0 \arrow[r] & \cdots \arrow[r] & C_k \\
 & Z \arrow[ul] \arrow[ur] \arrow[urr] \arrow[urrrr] & & & &
\end{tikzcd}$$

with the maps $A_i \to X$ acyclic fibrations and the maps $Z \to B_i$ and $Z \to C_j$ acyclic cofibrations. Write $P_\bullet$ for $\operatorname{diag} P_{\bullet\bullet}$. We have simplicial maps $\beta : P_\bullet \to N_\bullet\mathscr{C}$ and $\gamma : P_\bullet \to N_\bullet$ by forgetting about the $C_0, \ldots, C_k$ part of the diagram and the $B_0, \ldots, B_m$ part of the diagram respectively. These maps are easily seen to be weak equivalences. In addition, we have a map $\alpha : P_\bullet \to N_\bullet\mathscr{C}$ that sends the n-simplex of $P_\bullet$ pictured above (for $k = m = n$) to the n-simplex of $N_\bullet\mathscr{C}$ determined by the following diagram.

$$\begin{tikzcd}
 & X & \\
A_0 \arrow[ur] \arrow[r] \arrow[d] & \cdots \arrow[r] & A_n \arrow[ul] \arrow[d] \\
C_0 \arrow[r] & \cdots \arrow[r] & C_n \\
 & Z \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

The maps α and β are homotopic, and so α is also a weak equivalence. On the other hand, α is the composite of γ with the map $N_\bullet \to N_\bullet\mathscr{C}$ that we are interested in, which then must be a weak equivalence. □

8. The proof of Lemma 6.1

For the proof of Lemma 6.1, we need to show that the two inclusions into $L\mathscr{M}(X,Y_\bullet)$ are weak equivalences. It is elementary to see that the map $L\mathscr{M}(X,Y) \to \operatorname{diag} L\mathscr{M}(X,Y_\bullet)$ is a weak equivalence: Since each map $Y \to Y_n$ is a weak equivalence, each map $L\mathscr{M}(X,Y) \to L\mathscr{M}(X,Y_n)$ is a weak equivalence. The main difficulty is showing that the inclusion $\mathscr{M}(X,Y_\bullet) \to \operatorname{diag} L\mathscr{M}(X,Y_\bullet)$ is a weak equivalence. We argue by comparing this map with the constructions described in the last section.

To do this, we add another simplicial direction to the constructions; we need the special simplicial resolutions for the various Y_n's to be related by simplicial maps. In other words, we need to form from a simplicial resolution $Y \to Y_\bullet$ a kind of "bisimplicial resolution" $Y_\bullet \to Y_{\bullet\bullet}$. We prove the following lemma in the next section.

Lemma 8.1. *Let $Y_\bullet$ be a simpicial object in $\mathscr{M}$. There exists a bisimplicial object $Y_{\bullet\bullet}$ and a bisimplicial map from $Y_\bullet$ regarded as constant in the second direction to $Y_{\bullet\bullet}$ such that for each n, $Y_n \to Y_{n\bullet}$ is a special simplicial resolution* [4, 6.8]. *Moreover, if $Y_\bullet$ is a simplicial resolution then we can choose $Y_{\bullet\bullet}$ so that $Y_\bullet = Y_{\bullet 0}$ and the inclusion $Y_\bullet \to Y_{\bullet 0}$ is the identity.*

Now take $Z_\bullet$ to be $Y_{n\bullet}$. The simplicial sets $M_\bullet$ described in the previous section are natural in n and assemble to a bisimplicial set $(M_\bullet)_\bullet$. Write $\mathscr{M}_\bullet$ for the simplicial set $\mathscr{M}(X,Y_\bullet)$. We have a simplicial map $\mathscr{M}_\bullet \to (M_0)_\bullet$ induced by the identification $X^0 = X$, $Z_0 = Y_n$. We thereby obtain a simplicial map $\mathscr{M}_\bullet \to \operatorname{diag}(M_\bullet)_\bullet$.

Proposition 8.2. *The map $\mathscr{M}_\bullet \to \operatorname{diag}(M_\bullet)_\bullet$ is a weak equivalence.*

Proof. The composite map $\mathscr{M}_\bullet \to \operatorname{diag}(M_\bullet)_\bullet \to \operatorname{diag}\mathscr{M}(X^\bullet, Y_{\bullet\bullet})$ is the map induced by $X^\bullet \to X$ and $Y_\bullet \to Y_{\bullet\bullet}$ and is a weak equivalence by [4, 6.1–6.2]. The proof now follows from Proposition 7.2. □

The simplicial sets $L\mathscr{M}(X,Y_n)$ are natural in n and assemble to a bisimplicial set $L\mathscr{M}(X,Y_\bullet)$. However the constructed maps $(M_\bullet)_n \to L\mathscr{M}(X,Y_n)$ do not assemble to a bisimplicial map. We adjust for this difficulty as follows. Regarding $(M_\bullet)_\bullet$ as a functor from Δ^{op} to simplicial sets, we can form the two sided bar construction $M^h_\bullet = N_\bullet((M_\bullet)_\bullet, \Delta^{\mathrm{op}}, \Delta[\cdot])$. Here $\Delta[\cdot]$ denotes the functor from $(\Delta^{\mathrm{op}})^{\mathrm{op}} = \Delta$ that takes the object $\mathbf{n}$ to the standard simplicial n-simplex $\Delta[n]$. Regarding $\operatorname{diag}(M_\bullet)_\bullet$ as the coend of the functor $(M_\bullet)_\bullet \times \Delta[\cdot]$, we obtain a canonical map $M^h_\bullet \to \operatorname{diag}(M_\bullet)_\bullet$ that is easily seen to be a weak equivalence. Write $\mathscr{M}^h_\bullet$ for the two-sided bar construction $N_\bullet(\mathscr{M}_\bullet, \Delta^{\mathrm{op}}, \Delta[\cdot])$. The inclusion of $\mathscr{M}_\bullet$ as $(M_0)_\bullet$ induces a map $\mathscr{M}^h_\bullet \to M^h_\bullet$. The map

$\mathscr{M}^h_\bullet \to \mathscr{M}_\bullet$ is a weak equivalence, and so the map $\mathscr{M}^h_\bullet \to M^h_\bullet$ is a weak equivalence. We prove the following proposition.

Proposition 8.3. *There is a weak equivalence $M^h_\bullet \to \operatorname{diag} L\mathscr{M}(X, Y_\bullet)$ that makes the following diagram commute.*

$$\begin{tikzcd}
\mathscr{M}^h_\bullet \arrow[r, "\sim"] \arrow[d, "\sim"'] & \mathscr{M}_\bullet \arrow[d] \\
M^h_\bullet \arrow[r, "\sim"'] & \operatorname{diag} L\mathscr{M}(X, Y_\bullet)
\end{tikzcd}$$

Lemma 6.1 is an immediate consequence.

The map $M^h_\bullet \to L\mathscr{M}(X, Y_\bullet)$ is constructed by building homotopies between the simplicial structure maps on $(M_\bullet)_\bullet$ and those of $L\mathscr{M}(X, Y_\bullet)$. For example, consider a simplicial structure map in $Y_{\bullet\bullet}$ from $Y_{m\bullet}$ to $Y_{n\bullet}$. The corresponding simplicial structure map $(M_\bullet)_\bullet$ sends a 0-simplex $h : X^p \to Y_{mq}$ to the 0-simplex $X^p \to Y_{nq}$ given by the composite. The image of this in $L\mathscr{M}(X, Y_n)$ represented by the diagram on the left below. The corresponding simplicial structure map in $L\mathscr{M}(X, Y_\bullet)$ takes the image of h to the element of $L\mathscr{M}(X, Y_n)$ is represented by the diagram on the right below.

$$X \leftarrow X^p \to Y_{nq} \leftarrow Y_n \qquad X \leftarrow X^p \to Y_{mq} \leftarrow Y_m \to Y_n. \tag{8.4}$$

These diagrams do not represent the same element of $L\mathscr{M}(X, Y_n)$, although they are homotopic: the 1-simplex of $L\mathscr{M}(X, Y_n)$ represented by the diagram

$$\begin{tikzcd}
 & X^p \arrow[dl] \arrow[r] \arrow[d, "\mathrm{id}"'] & Y_{mq} \arrow[d] & Y_m \arrow[l] \arrow[d] \arrow[dr] & \\
X & X^p \arrow[l] \arrow[r] & Y_{nq} & Y_n \arrow[l] \arrow[r, "\mathrm{id}"'] & Y_n
\end{tikzcd} \tag{8.5}$$

has as its faces the two 0-simplexes represented by the diagrams above. We use this observation to build the necessary homotopies.

First, it is useful to factor the map $M_\bullet \to L\mathscr{M}(X, Z)$. Let $\mathscr{D}$ be the category whose objects are the maps $X^p \to Z_q$ and whose morphisms are the commuting diagrams

$$\begin{tikzcd}
X^p \arrow[r, "X(f)"] \arrow[d] & X^{p'} \arrow[d] \\
Z_q \arrow[r, "Z(g)"'] & Z_{q'}
\end{tikzcd}$$

for maps f in Δ and g in Δ^{op}. It is possible to show using the argument of Proposition 7.3 that the map $M_\bullet \to N_\bullet\mathscr{D}$ is a weak equivalence, but we shall not need this fact. The advantage of this factorization is that the map $N_\bullet\mathscr{D} \to L\mathscr{M}(X,Z)$ is induced by a functor. We can describe $L\mathscr{M}(X,Z)$ as the nerve of the category $\mathscr{E}$ whose objects are the "reduced hammocks of width zero and any length" between X and Z [3, 2.1] and whose maps are the reduced hammocks of width one and any length between X and Z (such a hammock gives a map from its first face to its zeroth face). The map $N_\bullet\mathscr{D} \to L\mathscr{M}(X,Z)$ is induced by the functor that takes the object $X^p \to Z_q$ to the hammock $X \leftarrow X^p \to Y_q \leftarrow Y$. Write $\mathscr{D}_n$ and $\mathscr{E}_n$ for the categories corresponding to $Z_\bullet = Y_{n\bullet}$. By construction, $\mathscr{D}_\bullet$ and $\mathscr{E}_\bullet$ are simplicial categories, but the functors $\mathscr{D}_n \to \mathscr{E}_n$ do not assemble to a simplicial functor.

Denote the functor $\mathscr{D}_n \to \mathscr{E}_n$ as F_n, and for g a map in Δ^{op} from $\mathbf{m} \to \mathbf{n}$, write D_g and E_g for the corresponding simplicial structure functors of $\mathscr{D}_\bullet$ and $\mathscr{E}_\bullet$. Consider the composite functors $E_g \circ F_m$ and $F_n \circ D_g$ from $\mathscr{D}_m$ to $\mathscr{E}_n$. These functors take the object $X^p \to Y_{mq}$ of $\mathscr{D}_m$ to the objects of $\mathscr{E}_n$ displayed in (8.4). The map in $\mathscr{E}_n$ displayed in (8.5) is a natural transformation τ_g from $E_g \circ F_m$ to $F_n \circ D_g$. Now let g' be a map in Δ^{op} from $\mathbf{k} \to \mathbf{m}$. We then have three composite functors $\mathscr{D}_k \to \mathscr{E}_n$

$$E_g \circ E_{g'} \circ F_k, \quad E_g \circ F_m \circ D_{g'}, \quad F_n \circ D_g \circ D_{g'},$$

related by natural transformations

$$\begin{tikzcd}
 & E_g \circ E_{g'} \circ F_k \arrow[dl, "E_g\tau_{g'}"'] \arrow[dr, "\tau_{g\circ g'}"] & \\
E_g \circ F_m \circ D_{g'} \arrow[rr, "\tau_g"'] & & F_n \circ D_g \circ D_{g'}.
\end{tikzcd}$$

Furthermore, it is elementary to see that the previous diagram commutes: the map $\tau_g \circ E_g\tau_{g'}$

$$\begin{tikzcd}
 & X^p \arrow[r] \arrow[dl] \arrow[d] & Y_{kq} \arrow[d] & Y_k \arrow[l] \arrow[r] \arrow[d] & Y_m \arrow[d] \arrow[dr] & \\
X & X^p \arrow[l] \arrow[r] \arrow[d] & Y_{mq} \arrow[d] & Y_m \arrow[l] \arrow[r] \arrow[d] & Y_m \arrow[r] \arrow[d] & Y_n \\
 & X^p \arrow[r] \arrow[ul] & Y_{nq} & Y_n \arrow[l] \arrow[r] & Y_n \arrow[ur] &
\end{tikzcd}$$

is easily seen to be the map $\tau_{g\circ g'}$ by the reduction rules [3, 2.1. (iv-v)$'$]. More generally for composable maps $\mathbf{n}_0 \xrightarrow{g_1} \cdots \xrightarrow{g_m} \mathbf{n}_m$ in Δ^{op}, we obtain a commuting m-simplex diagram. In other words, writing $g_{n,m}$ for the composite $g_n \circ \cdots \circ g_m$, in the following diagram

$$E_{g_{m,1}} \circ F_{n_0} \xrightarrow{E_{g_{m,2}}\tau_{g_1}} E_{g_{m,2}} \circ F_{n_1} \circ D_{g_1} \xrightarrow{E_{g_{m,3}}\tau_{g_2}} \cdots \xrightarrow{\tau_{g_m}} F_{n_m} \circ D_{g_{m,1}},$$

the composite map $E_{g_{m,j}} \circ F_{n_{j-1}} \circ D_{g_{j-1,1}} \to E_{g_{m,k}} \circ F_{n_k} \circ D_{g_{k-1,1}}$ for $j<k$ is the map $E_{g_{m,j+1}}\tau_{g_{k-1},j}$.

Let $\mathscr{F}^m$ denote the category with $m+1$ objects labeled $\mathbf{0},\dots,\mathbf{m}$, and a unique map $\mathbf{j}\to\mathbf{k}$ whenever $j\le k$. Then from the work above, composable maps $\mathbf{n}_0 \xrightarrow{g_1} \cdots \xrightarrow{g_m} \mathbf{n}_m$ in Δ^{op} induce a functor $\mathscr{D}_{n_0}\times\mathscr{F}^m \to \mathscr{E}_{n_m}$. Denote this functor as $F_{g_1,\dots,g_m}$. For fixed n, let $\mathscr{C}_n$ be the category obtained as the disjoint union of the categories $\mathscr{D}_{n_0}\times\mathscr{F}^m$ over the composable maps $\mathbf{n}_0 \xrightarrow{g_1} \cdots \xrightarrow{g_m} \mathbf{n}_m$ and maps $f:\mathbf{n}_m\to\mathbf{n}$ in Δ^{op} (m, n_0, n_m varying)

$$\mathscr{C}_n = \coprod_{\substack{\mathbf{n}_0 \xrightarrow{g_1}\cdots\xrightarrow{g_m}\mathbf{n}_m,\\ m,\, f:\mathbf{n}_m\to\mathbf{n}}} \mathscr{D}_{n_0}\times\mathscr{F}^m.$$

Then $E_f\circ F_{g_1,\dots,g_m}$ defines a functor $G_n:\mathscr{C}_n\to\mathscr{E}_n$.

We can assemble the $\mathscr{C}_n$ into a simplicial category in the evident way, defining faces and degeneracies by the functors

$$d_i:(\mathscr{D}_{n_0}\times\mathscr{F}^m, g_1,\dots,g_m,f)\mapsto(\mathscr{D}_{n_0}\times\mathscr{F}^m, g_1,\dots,g_m,d_i\circ f),$$

$$s_i:(\mathscr{D}_{n_0}\times\mathscr{F}^m, g_1,\dots,g_m,f)\mapsto(\mathscr{D}_{n_0}\times\mathscr{F}^m, g_1,\dots,g_m,s_i\circ f).$$

The functors G_n then clearly assemble to a simplicial functor $G_\bullet:\mathscr{C}_\bullet\to\mathscr{E}_\bullet$. Applying the nerve, we obtain a simplicial map $N_\bullet\mathscr{C}_\bullet\to N_\bullet\mathscr{E}_\bullet$.

Since the nerve of $\mathscr{F}^m$ is canonically isomorphic to $\Delta[m]$, and the set of maps $f:\mathbf{n}_m\to\mathbf{n}$ in Δ^{op} is the set of n-simplexes of $\Delta[n_m]$, the nerve of $\mathscr{C}_\bullet$ is easily seen to be the simplicial set

$$N_\bullet\mathscr{C}_\bullet \cong \coprod_m N_m(N_\bullet\mathscr{D}_\bullet, \Delta^{\mathrm{op}}, \Delta[\cdot])\times\Delta[m].$$

There is a canonical quotient map from the righthand side above to the two-sided bar construction $N_\bullet(N_\bullet\mathscr{D}_\bullet,\Delta^{\mathrm{op}},\Delta[\cdot])$. We show that the map $N_\bullet\mathscr{C}_\bullet\to N_\bullet\mathscr{E}_\bullet$ factors through this quotient as follows.

Note that $\mathscr{F}^\bullet$ is a cosimplicial category, and let $\mathscr{B}^\bullet_{\bullet\bullet}$ be the simplicial cosimplicial simplicial category

$$\mathscr{B}^m_{k,n} = \coprod_{\substack{\mathbf{n}_o \xrightarrow{g_1}\cdots\xrightarrow{g_k}\mathbf{n}_k,\\ f:\mathbf{n}_k\to\mathbf{n}}} \mathscr{D}_{n_0}\times\mathscr{F}^m.$$

For $k=m$, we can regard $\mathscr{B}^m_{m,n}$ as a subcategory of $\mathscr{C}_n$. For $e:\mathbf{k}\to\mathbf{m}$ in Δ^{op}, we have functors

$$B^e:\mathscr{B}^m_{k,n}\to\mathscr{B}^k_{k,n} \quad\text{and}\quad B_e:\mathscr{B}^m_{k,n}\to\mathscr{B}^m_{m,n}.$$

Consider the composite functors

$$\mathscr{B}^m_{k,n}\to\mathscr{B}^k_{k,n}\to\mathscr{C}_n\to\mathscr{E}_n \quad\text{and}\quad \mathscr{B}^m_{k,n}\to\mathscr{B}^m_{m,n}\to\mathscr{C}_n\to\mathscr{E}_n.$$

It is straightforward to check from the construction of G_n that these two functors coincide. We can identify the nerve of $\mathscr{B}^k_{m,\bullet}$ as

$$N_\bullet \mathscr{B}^k_{m,\bullet} \cong \coprod_{k,m} N_k(N_\bullet \mathscr{D}_\bullet, \Delta^{\mathrm{op}}, \Delta[\cdot]) \times \Delta[m],$$

and it follows that the map $N_\bullet G_\bullet$ factors through the coequalizer of

$$\coprod_{h:\mathbf{k}\to\mathbf{m}} N_k(N_\bullet \mathscr{D}_\bullet, \Delta^{\mathrm{op}}, \Delta[\cdot]) \times \Delta[m] \rightrightarrows \coprod_m N_m(N_\bullet \mathscr{D}_\bullet, \Delta^{\mathrm{op}}, \Delta[\cdot]) \times \Delta[m],$$

which is the two-sided bar construction $N_\bullet(N_\bullet \mathscr{D}_\bullet, \Delta^{\mathrm{op}}, \Delta[\cdot])$.

Let $\mu : M^h_\bullet \to \operatorname{diag} L\mathscr{M}(X, Y_\bullet)$ be the map

$$\begin{aligned} M^h_\bullet &= N_\bullet((\mathscr{M}_\bullet)_\bullet, \Delta^{\mathrm{op}}, \Delta[\cdot]) \\ &\to N_\bullet(N_\bullet \mathscr{D}_\bullet, \Delta^{\mathrm{op}}, \Delta[\cdot]) \to N_\bullet \mathscr{E}_\bullet = \operatorname{diag} L\mathscr{M}(X, Y_\bullet). \end{aligned}$$

To see that μ is a weak equivalence, we note that the composite of the inclusion of $(M_\bullet)_0$ in $M^h_\bullet$ and the map μ factors through the weak equivalence $(M_\bullet)_0 \to L\mathscr{M}(X, Y_0)$ described in Section 7.

It is easily checked that both composite maps $\mathscr{M}^h_\bullet \to \operatorname{diag} L\mathscr{M}(X, Y_\bullet)$ send a general element $(h, g_1, \ldots, g_m, f)$ of $\mathscr{M}^h_n$ specified by a map $h : X \to Y_{n_0}$ in $\mathscr{M}$ and maps $\mathbf{n}_0 \xrightarrow{g_1} \cdots \xrightarrow{g_m} \mathbf{n}_m$ and $f : \mathbf{n}_m \to \mathbf{n}$ in Δ^{op} to the element of $L\mathscr{M}(X, Y_n)$ specified by (the reduction of) the diagram

$$X \xleftarrow{\mathrm{id}} X^0 \xrightarrow{Y(f \circ g_{m,1}) \circ h} Y_{n0} \xleftarrow{\mathrm{id}} Y_n.$$

This implies that the diagram in Proposition 8.3 commutes.

9. The proof of Lemma 8.1

The proof of Lemma 8.1 uses the closed model category structures constructed in [9]. The work of [9] shows that for any closed model category $\mathscr{M}$, the category $\Delta^{\mathrm{op}}\mathscr{M}$ of simplicial objects in $\mathscr{M}$ is a closed model category with weak equivalence the level-wise weak equivalences, those simplicial maps $X_\bullet \to Y_\bullet$ that restrict to weak equivalence in each degree $X_n \to Y_n$.

The cofibrations and fibrations are defined using the "skeleton" and "coskeleton" functors. Denote by Δ^{op}_n the full subcategory of Δ^{op} consisting of the objects $\mathbf{1}, \ldots, \mathbf{n}$ and denote by I_n the inclusion functor. The n-skeleton functor sk_n and the n-coskeleton functor ck_n are defined as the left and right Kan extensions of I_n [7, Section 3] respectively. Since $\mathscr{M}$ has finite colimits and limits, these functors exist by [7, 3.1]. This description makes it clear that $\mathrm{sk}_n Y_\bullet$ and $\mathrm{ck}_n Y_\bullet$ are simplicial objects, that the canonical maps $s_n : \mathrm{sk}_n Y_\bullet \to Y_\bullet$ and $c_n : Y_\bullet \to \mathrm{ck}_n Y_\bullet$ are simplicial maps, and that s_n and c_n are isomorphisms in degrees less than $n+1$, but there are more illuminating inductive descriptions of sk_n and ck_n; the diagram on the left below is a pushout and the diagram on the right is a pullback in $\mathscr{M}$

$$
\begin{array}{ccc}
\coprod_{s:\mathbf{n}\to\mathbf{k}} (\mathrm{sk}_{n-1}Y_\bullet)_n & \longrightarrow & \coprod_{s:\mathbf{n}\to\mathbf{k}} Y_n \\
\downarrow & & \downarrow \\
(\mathrm{sk}_{n-1}Y_\bullet)_k & \longrightarrow & (\mathrm{sk}_n Y_\bullet)_k
\end{array}
\qquad
\begin{array}{ccc}
(\mathrm{ck}_n Y_\bullet)_k & \longrightarrow & (\mathrm{ck}_{n-1}Y_\bullet)_k \\
\downarrow & & \downarrow \\
\prod_{d:\mathbf{k}\to\mathbf{n}} Y_n & \longrightarrow & \prod_{d:\mathbf{k}\to\mathbf{n}} (\mathrm{ck}_{n-1}Y_\bullet)_n
\end{array}
\tag{9.1}
$$

where the coproduct is over the degeneracy maps $\mathbf{n}\to\mathbf{k}$ (the maps in Δ^{op} opposite to the surjections $\mathbf{k}\to\mathbf{n}$) and the product is over the face maps $\mathbf{k}\to\mathbf{n}$ (the maps in Δ^{op} opposite to the injections $\mathbf{n}\to\mathbf{k}$).

The fibrations in $\Delta^{\mathrm{op}}\mathscr{M}$ are defined to be those simplicial maps $X_\bullet\to Y_\bullet$ for which the induced map

$$X_n \to (\mathrm{ck}_{n-1}X)_n \times_{(\mathrm{ck}_{n-1}Y)_n} Y_n \tag{9.2}$$

is a fibration in $\mathscr{M}$ for each n. It turns out that a simplicial map $X_\bullet\to Y_\bullet$ is an acyclic fibration if and only if the map (9.2) is an acyclic fibration for each n [9, 1.4.b]. Similarly a simplicial map $X_\bullet\to Y_\bullet$ is a cofibration or acyclic cofibration in $\Delta^{\mathrm{op}}\mathscr{M}$ if and only if the map

$$(\mathrm{sk}_{n-1}Y)_n \amalg_{(\mathrm{sk}_{n-1}X)_n} X_n \to Y_n \tag{9.3}$$

is a cofibration or acyclic cofibration in $\mathscr{M}$ for each n [9, 1.4.a]. One useful fact about this model structure is the following.

Proposition 9.4. *If $X_\bullet\to Y_\bullet$ is a cofibration, fibration, or weak equivalence then so is each map $X_n\to Y_n$.*

Proof. The case of a weak equivalence follows from the definition. For the case of a cofibration assume by induction that the map $\mathrm{sk}_{n-1}X_\bullet\to\mathrm{sk}_{n-1}Y_\bullet$ is a level-wise cofibration. Then we can write $(\mathrm{sk}_nX_\bullet)_k\to(\mathrm{sk}_nY_\bullet)_k$ as the composite of the cofibration

$$
\begin{aligned}
(\mathrm{sk}_nX_\bullet)_k &= \left(\coprod X_n\right) \amalg_{(\amalg(\mathrm{sk}_{n-1}X_\bullet)_n)} (\mathrm{sk}_{n-1}X_\bullet)_k \\
&\to \left(\coprod X_n\right) \amalg_{(\amalg(\mathrm{sk}_{n-1}X_\bullet)_n)} (\mathrm{sk}_{n-1}Y_\bullet)_k
\end{aligned}
$$

and the cofibration obtained from (9.3) by cobase change

$$
\begin{aligned}
&\left(\coprod X_n\right) \amalg_{(\amalg(\mathrm{sk}_{n-1}X_\bullet)_n)} (\mathrm{sk}_{n-1}Y_\bullet)_k \\
&\cong \left(\left(\coprod X_n\right) \amalg_{(\amalg(\mathrm{sk}_{n-1}X_\bullet)_n)} \left(\coprod(\mathrm{sk}_{n-1}Y_\bullet)_n\right)\right) \amalg_{(\amalg(\mathrm{sk}_{n-1}Y_\bullet)_n)} (\mathrm{sk}_{n-1}Y_\bullet)_k
\end{aligned}
$$

$$\rightarrow \left(\coprod Y_n\right) \amalg_{(\amalg(\mathrm{sk}_{n-1}Y_\bullet)_n)} (\mathrm{sk}_{n-1}Y_\bullet)_k$$
$$= (\mathrm{sk}_n Y_\bullet)_k.$$

It follows in particular the the map $X_n = (\mathrm{sk}_n X_\bullet)_n$ to $Y_n = (\mathrm{sk}_n Y_\bullet)_n$ is a cofibration. The case of a fibration is entirely similar. □

It is straightforward to identify $(\mathrm{ck}_n Y_\bullet)_{n+1}$ as the object (d_*, Y_n) of [4, 4.3] (denoted as L_n^{n+1} in Section 6). It follows that $Y \to Y_\bullet$ is a simplicial resolution in the sense of [4, 4.3] if and only if $Y \to Y_\bullet$ is a weak equivalence in $\Delta^{\mathrm{op}}\mathscr{M}$ and $Y_\bullet$ is fibrant in $\Delta^{\mathrm{op}}\mathscr{M}$. Likewise since $(\mathrm{sk}_n Y_\bullet)_{n+1}$ is the object (s_*, Y_n) of [4, 6.7], the "special simplicial resolutions" of [4, 6.8] are the fibrant approximations $Y \to Y_\bullet$ in $\Delta^{\mathrm{op}}\mathscr{M}$.

Proof of Lemma 8.1. Given $Y_\bullet$, we choose $Y_\bullet \to Y_{\bullet\bullet}$ to be a special simplicial resolution in the closed model category $\Delta^{\mathrm{op}}\mathscr{M}$. The construction given in [4, 6.7] shows that if $Y_\bullet$ is fibrant in $\Delta^{\mathrm{op}}\mathscr{M}$, we can choose $Y_{\bullet 0} = Y_\bullet$. The proposition follows from the fact that (acyclic) cofibrations and fibrations in $\Delta^{\mathrm{op}}\mathscr{M}$ are level-wise (acyclic) cofibrations and fibrations. □

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